Journal of Applied Mechanics and Technical Physics, Vol. 37, No. 5, 1996

PLANE WAVES AND FUNDAMENTAL SOLUTIONS IN LINEAR THERMOELECTROELASTICITY

A. O. Vatul'yan, A. Yu. Kiryutenko, and A. V. Nasedkin

UDC 539.3:541

The behavior of dielectric media with allowance for pyroelectric and piezoelectric effects within the framework of a linear quasi-electrostatic approximation is described by the theory of thermoelectroelasticity (thermopiezoelectricity) [1]. The subsequent development of this theory is reflected in [2, 3]. A small number of papers are devoted to the properties of associate thermoelectroelastic waves. Dispersion relations for a series of semi-confined thermoelectroelastic media were analyzed in [4, 5].

In this paper, the structure of plane waves in an unconfined thermoelectroelastic medium of 6 mm class is investigated. Dimensionless parameters that reflect the associations of mechanical, electrical, and thermal fields are determined. The effect of associating on the velocities and damping factors of modified electroelastic and thermal waves is studied. Dependences of dispersion properties of plane waves on frequency of vibrations and direction of their propagation are analyzed. Numerical calculations are performed for the concrete thermoelectroelastic medium of barium titanate ($BaTiO_3$). Fundamental solutions of the two-dimensional problem of linear thermoelectroelasticity for a medium of 6 mm class are studied. Representations in the form of simple integrals over a finite interval are constructed which are convenient for implementation of the boundary element method.

1. Analysis of Plane Waves in an Unconfined Thermoelectroelastic Medium. Let us consider the equations of motion of a thermoelectroelastic medium for piezoceramics polarized along the Ox_3 axis (6 mm class) [3]:

$$L_{ij}U_j = 0. \tag{1.1}$$

Here $\mathbf{U} = \{u_1, u_2, u_3, \varphi, \theta\}$ is the vector of unknowns (u_j are the components of the displacement vector, φ is the potential, and θ is the temperature increase above a natural state), and L_{ij} are the partial differential operators defined by the formulas

$$\begin{gathered}
L_{11} = c_{11}\partial_1^2 + 0.5(c_{11} - c_{12})\partial_2^2 + c_{44}\partial_3^2 - \rho\partial_t^2, \quad L_{12} = L_{21} = 0.5(c_{11} + c_{12})\partial_1\partial_2, \\
L_{13} = L_{31} = (c_{13} + c_{44})\partial_1\partial_3, \quad L_{14} = L_{41} = (e_{31} + e_{15})\partial_1\partial_3, \quad L_{15} = -\gamma_{11}\partial_1, \\
L_{51} = T_0\gamma_{11}\partial_1\partial_t, \quad L_{22} = 0.5(c_{11} - c_{12})\partial_1^2 + c_{11}\partial_2^2 + c_{44}\partial_3^2 - \rho\partial_t^2, \\
L_{23} = L_{32} = (c_{13} + c_{44})\partial_2\partial_3, \quad L_{24} = L_{42} = (e_{31} + e_{15})\partial_2\partial_3, \\
L_{25} = -\gamma_{11}\partial_2, \quad L_{52} = T_0\gamma_{11}\partial_2\partial_t, \quad L_{33} = c_{44}(\partial_1^2 + \partial_2^2) + c_{33}\partial_3^2 - \rho\partial_t^2, \\
L_{34} = L_{43} = e_{15}(\partial_1^2 + \partial_2^2) + e_{33}\partial_3^2, \quad L_{35} = -\gamma_{33}\partial_3, \qquad L_{53} = T_0\gamma_{33}\partial_3\partial_t, \\
L_{44} = -(\text{э}_{11}(\partial_1^2 + \partial_2^2) + \text{э}_{33}\partial_3^2), \quad L_{45} = g_3\partial_3, \quad L_{54} = -T_0 g_3\partial_3\partial_t, \\
L_{55} = (\rho c_\varepsilon\partial_t - k_{11}\partial_1^2 - k_{11}\partial_2^2 - k_{33}\partial_3^2),
\end{gathered} \tag{1.2}$$

where c_{ij} are the elastic moduli, e_{ij} are the piezoelectric moduli, э_{ij} are the dielectric permittivities, γ_{ij} are the thermal-stress coefficients, g_i are the pyroelectric coefficients, k_{ij} are the thermal conductivity coefficients, c_ε is the heat capacity, ρ is the density, and T_0 is the natural state temperature in the Kelvin scale.

Rostov State University, Rostov-on-Don 344090. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 37, No. 5, pp. 135–142, September–October, 1996. Original article submitted July 25, 1995.

0021-8944/96/3705-0727 $15.00

We make system (1.1) nondimensional by introducing the following dimensionless parameters and variables:

$$\tilde{c}_{ij} = c_{ij}/c_{33}, \quad \tilde{u}_j = u_j/h, \quad \tilde{\partial}_j = h\partial_j, \quad \tilde{x}_j = x_j/h, \quad \tilde{t} = t/t_0, \quad \tilde{\partial}_t = t_0\partial_t,$$

$$\tilde{\varphi} = \sqrt{\mathfrak{z}_{33}/c_{33}}\varphi/h, \quad \tilde{e}_{ij} = e_{ij}/\sqrt{\mathfrak{z}_{33}c_{33}}, \quad \tilde{\rho} = \rho/\rho_0, \quad \tilde{\theta} = \theta/\theta_0, \quad \tilde{\mathfrak{z}}_{ij} = \mathfrak{z}_{ij}/\mathfrak{z}_{33},$$

$$\tilde{\gamma}_{ij} = \gamma_{ij}\sqrt{T_0/(c_{33}\rho c_\epsilon)}, \quad \tilde{g}_3 = g_3\sqrt{T_0/(\mathfrak{z}_{33}\rho c_\epsilon)}.$$

Next, assuming that $h^2\rho/(t_0^2 c_{33}) = 1$ and $k_{33}t_0/(\rho c_\epsilon h^2) = 1$ and setting $\rho_0 = \rho$, $æ = k_{33}/(\rho c_\epsilon)$, $v_0^2 = c_{33}/\rho$, and $\theta_0 = \sqrt{T_0 c_{33}/(\rho c_\epsilon)}$, we obtain $t_0 = æ/v_0^2$ and $h = æ/v_0$.

The chosen method of nondimensionalization reduces the system of differential equations (1.1) to a form that is convenient for the subsequent analysis and finding the parameters of fields associating. Thus, the coefficients of $\partial^2\tilde{U}_j/\partial\tilde{t}^2$ and $\partial\tilde{\theta}/\partial\tilde{t}$ are equal to unity, and the coefficients of $\partial^2\tilde{U}_j/\partial\tilde{x}_m^2$ on the order of unity. The quantities $\tilde{e}_{i\alpha}$ determine electromechanical relations and are on the order $k^2/(1-k^2)$, where k^2 is one of the static coefficients of an electromechanical relation (CEMR) [6]. Since $k^2 < 0.5$ for real piezoelectric media, we have $0 < \tilde{e}_{i\alpha}^2 < 1$. The quantities $\tilde{\gamma}_{ij}$ define associating of elastic and thermal fields. Moreover, $\tilde{\gamma}_{jj}^2$ are completely analogous to associating constants in thermoelasticity problems [7] and are small for most media. Finally, $\tilde{g}_3$ reflects the associating of electric and thermal fields. From the positive definiteness of internal energy it follows that for a piezoelectric medium of 6 mm class the inequality $\mathfrak{z}_{33}\rho c_\epsilon/T_0 \geqslant g_3^2$ holds and, hence, $\tilde{g}_3^2 < 1$. We note also that the time and spatial characteristic parameters t_0 and h and the quantity $æ$ are usual with nondimensionalizing of the thermoelasticity equations [7], and v_0 has the meaning of the characteristic velocity of acoustical waves in a piezoelectric medium. Below we shall omit the tilde sign above dimensionless quantities.

We investigate plane waves in an unconfined thermoelectroelastic medium, i.e., we seek a solution of nondimensionalized equations (1.1) in the form

$$U_j = X_j \exp[i(\omega t - \eta \mathbf{n}\mathbf{x})], \tag{1.3}$$

where ω is the dimensionless real frequency of vibrations related to the dimensional frequency Ω by the formula $\omega = \Omega/\Omega_*$ $[\Omega_* = c_{33}/(\rho æ)]$, $\mathbf{n}$ is a unit vector that determines the direction of wave propagation, and η is a generalized (in general, complex) dimensionless wave number.

Substituting (1.3) into (1.2) and equating the determinant of the resulting algebraic system to zero, we obtain a dispersion relation between ω, η, and $\mathbf{n}$. The set of roots of these equations is divided into two subsets. One of them characterizes a non-associated, purely elastic SH-wave that is polarized in the plane Ox_1x_2 and not subjected to dispersion and damping; its velocity is $v_4 = \omega/\eta = \sqrt{0.5(c_{11}-c_{12})(n_1^2+n_2^2) + c_{44}n_3^2}$, where n_1, n_2, and n_3 are the components of the vector $\mathbf{n}$. The second subset coincides with the set of zeros of the determinant $D(\omega,\eta) = \eta^2 D_0(\omega,\eta)$ for fixed $\mathbf{n}$.

Let us investigate the root set structure of the equation

$$D_0(\omega,\eta) = 0. \tag{1.4}$$

If we put $z = \eta/\omega$, Eq. (1.4) is written as

$$\begin{vmatrix} c_1 z^2 - 1 & c_0 z^2 & e_0 z & \gamma_1 \alpha z \\ c_0 z^2 & c_3 z^2 - 1 & e_3 z & \gamma_3 \beta z \\ e_0 z & e_3 z & -\mathfrak{z} & -g\beta \\ \gamma_1 \alpha z & \gamma_3 \beta z & -g\beta & iK\omega z^2 - 1 \end{vmatrix} = 0. \tag{1.5}$$

Here $\alpha = \sqrt{n_1^2+n_2^2} = \cos\psi$, $\beta = n_3 = \sin\psi$, $c_1 = c_{11}\alpha^2 + c_{44}\beta^2$, $c_0 = (c_{13}+c_{44})\alpha\beta$, $e_0 = (e_{31}+e_{15})\alpha\beta$, $c_3 = c_{44}\alpha^2 + c_{33}\beta^2$, $e_3 = e_{15}\alpha^2 + e_{33}\beta^2$, $\mathfrak{z} = \mathfrak{z}_{11}\alpha^2 + \mathfrak{z}_{33}\beta^2$, $K = k_{11}/k_{33}\alpha^2 + \beta^2$, $\gamma_i = \gamma_{ii}$, and $g = g_3$. Introducing $\bar{g}$ and $\bar{\gamma}_j$ by the formulas $g = \varepsilon\bar{g}$ and $\gamma_j = \varepsilon\bar{\gamma}_j$, from (1.5) after certain transformations we obtain

$$iK\omega z^2(a_1 z^4 - a_2 z^2 + a_3) - (A_1 z^4 - A_2 z^2 + A_3) = 0, \tag{1.6}$$

where $A_j = a_j + \varepsilon^2 q_j$ ($j = 1$, 2, and 3); $q_1 = b_1 - d_1 - f_1$; $q_2 = b_2 - d_2 - f_2$; $q_3 = -d_3$; $a_1 = -\ni c_1 c_3 + 2c_0 e_0 e_3 - e_0^2 c_3 + c_0^2 \ni - e_3^2 c_1$; $a_2 = -(\ni(c_1 + c_3) + e_0^2 + e_3^2)$; $a_3 = -\ni$; $b_1 = -2\bar{g}\beta[\bar{\gamma}_3\beta(c_0 e_0 - e_3 c_1) + \bar{\gamma}_1\alpha(c_0 e_3 - e_0 c_3)]$; $b_2 = 2\bar{g}\beta[e_0\bar{\gamma}_1\alpha + e_3\bar{\gamma}_3\beta]$; $d_1 = -\bar{g}^2\beta^2(c_1 c_3 - c_0^2)$; $d_2 = -\bar{g}^2\beta^2(c_1 + c_3)$; $d_3 = -\bar{g}^2\beta^2$; $f_1 = \bar{\gamma}_1^2\alpha^2(\ni c_3 + e_3^2) + \bar{\gamma}_3^2\beta^2(c_0^2 + c_1\ni) - 2\bar{\gamma}_1\bar{\gamma}_3\alpha\beta(e_0 e_3 + \ni c_0)$; and $f_2 = \ni(\bar{\gamma}_1^2\alpha^2 + \bar{\gamma}_3^2\beta^2)$.

Equation (1.6) has six real roots z of which only three roots z_j ($j = 1$, 2, and 3) with $\operatorname{Im} z_j > 0$ ($\operatorname{Im}\eta_j > 0$ and $\eta_j = z_j\omega$) will be analyzed.

For $\varepsilon = 0$ we have a non-associated problem of thermoelectroelasticity, and we find the roots z_j in explicit form:

$$z_1^2 = 1/(iK\omega), \quad z_2^2 = (a_2 + r)/(2a_1), \quad z_3^2 = (a_2 - r)/(2a_1), \quad r = \sqrt{a_1^2 - 4a_1 a_3}. \tag{1.7}$$

The first root characterizes a purely thermal wave with velocity $v_1 = \operatorname{Re}(1/z_1)$ and damping $\nu_1 = -\operatorname{Im}(\omega z_1)$. The second and third roots characterize electroelastic waves that are not subjected to dispersion and a damping. The velocities of these waves $v_i = \operatorname{Re}(1/z_i)$ ($i = 2$ and 3) are identical to those given in [8].

Setting $Y = z^2$, we rewrite Eq. (1.6) as

$$(iK\omega Y - 1)(a_1 Y^2 - a_2 Y + a_3) - \varepsilon^2(q_1 Y^2 - q_2 Y + q_3) = 0. \tag{1.8}$$

Taking into account that the parameter ε is small for real piezoelectric media, we seek the roots Y_j of Eq. (1.8) in the form of an expansion in powers of the parameter ε:

$$Y_j = y_{j0} + \varepsilon^2 y_{j1} + \dots$$

The first terms of this expansion are given by the formulas

$$y_{j0} = z_j^2, \quad y_{11} = (q_1 y_0^2 - q_2 y_0 + q_3)/[iK\omega(a_1 y_0^2 - a_2 y_0 + a_3)],$$

$$y_{j1} = (q_1 y_0^2 - q_2 y_0 + q_3)/[(iK\omega y_0 - 1)(2a_1 y_0 - a_2)], \quad j = 2, 3$$

[z_j are given in (1.7)].

For the associated problem for $\varepsilon \neq 0$ we have a modified quasi-thermal wave ($j = 1$) and two modified quasi-electroelastic waves ($j = 2$ and 3) subjected to damping and dispersion. It should be noted that the acoustical frequencies ω for which the theory of thermoelectroelasticity is substantiated are within the limits $\omega \ll 1$ ($\Omega \ll \Omega_*$). In this connection it is also important to consider the asymptotic behavior of the roots $X_j = 1/z_j^2$ for small ω which are determined from (1.8) in the form

$$X_j = x_{j0} + \omega x_{j1} + \omega^2 x_{j2} + \dots, \quad j = 1, 2, \text{ and } 3,$$

where $x_{10} = 0$; $x_{11} = iKa_1/A_1$; $x_{12} = x_{11}(A_2 x_1 - iKa_2)/A_1$; $x_{20} = (A_2 + r_1)/(2A_3)$; $r_1 = \sqrt{A_2^2 - 4A_3A_1}$; $x_{30} = (A_2 + r_1)/(2A_3)$; $x_{k1} = iK(a_1 - a_2 x_{k0} + a_3 x_{k0}^2)/[x_{k0}(2A_3 x_{k0} - A_2)]$ ($k = 2$ and 3); and $x_{k2} = x_{k1}[iK(2a_3 x_{k0} - a_2) - x_{k1}(3A_3 x_{k0} - A_2)]/[x_{k0}(2A_3 x_{k0} - A_2)]$.

Using methods of perturbation theory, we can construct the dependences of the velocities and dampings of thermoelectroelastic waves on the associating coefficient ε for the directions $\psi = 0$ and $\psi = \pi/2$:

(1) For $\psi = 0$

$$v_1 = \sqrt{0.5k_m\omega}\Big(1 - \varepsilon^2\frac{\bar{\gamma}_1^2(c_{11} - k_m\omega)}{2(c_{11}^2 + k_m^2\omega^2)}\Big), \quad \nu_1 = \sqrt{0.5k_m\omega}\Big(1 - \varepsilon^2\frac{\bar{\gamma}_1^2(c_{11} + k_m\omega)}{2(c_{11}^2 + k_m^2\omega^2)}\Big),$$

$$v_2 = \sqrt{c_{11}}\Big(1 + \varepsilon^2\frac{\bar{\gamma}_1{}^2 c_{11}}{2(c_{11}^2 + k_m^2\omega^2)}\Big), \quad \nu_2 = \varepsilon^2\frac{\bar{\gamma}_1^2\sqrt{c_{11}}k_m\omega}{2(c_{11}^2 + k_m^2\omega^2)},$$

$$v_3 = \sqrt{c_{44}\Big(1 + \frac{e_{15}^2}{c_{44}\ni_{11}}\Big)}, \quad \nu_3 = 0;$$

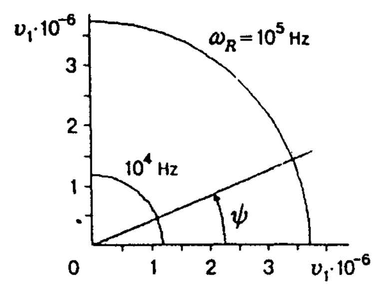

Fig. 1

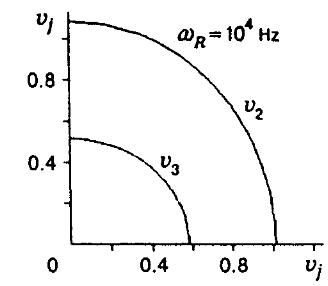

Fig. 2

(2) For $\psi = \pi/2$

$$v_1 = \sqrt{\frac{\omega}{2(1-\varepsilon^2\bar{g}^2)}}\left(1 - \varepsilon^2 \frac{(\bar{\gamma}_3^2 - 2\bar{\gamma}_3\bar{g}e_{33} + \bar{g}^2e_{33}^2)(c_{33} + e_{33}^2 - \omega)}{2(\omega^2 + c_{33} + e_{33}^2)}\right),$$

$$\nu_1 = \sqrt{\frac{\omega}{2(1-\varepsilon^2\bar{g}^2)}}\left(1 - \varepsilon^2 \frac{(\bar{\gamma}_3^2 - 2\bar{\gamma}_3\bar{g}e_{33} + \bar{g}^2e_{33}^2)(c_{33} + e_{33}^2 + \omega)}{2(\omega^2 + c_{33} + e_{33}^2)}\right),$$

$$v_2 = \sqrt{\frac{c_{33} + e_{33}^2}{1 - \varepsilon^2\bar{g}^2}}\left(1 + \varepsilon^2 \frac{(\bar{\gamma}_3^2 - 2\bar{\gamma}_3\bar{g}e_{33} - \bar{g}^2c_{33})(c_{33} + e_{33}^2) - \bar{g}^2\omega^2}{2(\omega^2 + c_{33} + e_{33}^2)}\right),$$

$$\nu_2 = \sqrt{\frac{c_{33} + e_{33}^2}{1 - \varepsilon^2\bar{g}^2}}\left(\varepsilon^2\omega \frac{\bar{\gamma}_3^2 - 2\bar{\gamma}_3\bar{g}e_{33} + \bar{g}^2e_{33}^2}{2(\omega^2 + c_{33} + e_{33}^2)}\right), \quad v_3 = \sqrt{c_{44}}, \quad \nu_3 = 0.$$

Here $k_m = k_{11}/k_{33}$; $g = \varepsilon\bar{g}$, and $\gamma_j = \varepsilon\bar{\gamma}_j$ ($j = 1$ and 3).

These formulas enable us to analyze the dependences of the velocities and dampings on associating of the problem and the frequency of vibrations. We note that the structure of the dispersion equations (1.6) and (1.8) and the properties of plane waves are similar in many respects to those in the case of a thermoelastic transversely isotropic medium considered in [9, 10].

Figures 1–5 show the propagation and damping velocities of modified thermal and electroelastic waves for barium titanate versus the propagation direction characterized by the polar angle ψ for various $\omega = \omega_R$, temperature of the undisturbed state $T_0 = 300$ K, and $\varepsilon = 0.01$. Physical constants are the same as in [11].

A simple analysis leads to the following conclusions: the velocities of quasi-electroelastic waves are practically independent of the frequency of vibrations [one of these waves will be named the quasi-longitudinal wave (v_2 and ν_2) and the second the quasi-transverse wave (v_3 and ν_3)]. The dampings of these waves depend significantly on the frequency of vibrations. Moreover, the damping of the quasi-longitudinal wave depends weakly on the direction of its propagation, and the damping of the quasi-transverse wave depends significantly on the angle ψ. Note also that the quasi-transverse wave becomes purely transverse and undamped for the directions $\psi = 0$ and $\psi = \pi/2$. The velocity and damping of the quasi-thermal wave depend significantly on the frequency of vibrations (as in the thermoelasticity problems in [7]).

2. Construction of Fundamental Solutions for a Two-Dimensional Thermoelectroelasticity Problem. The question on the construction of fundamental solutions in linear thermoelectroelasticity can be solved on the basis of analysis of the zeros of $D_0(\omega, \eta)$. This question is important for applications, in particular, for the formulation of the boundary integral equations and the implementation of boundary element method. We confine our attention to the case of plane deformation of a 6 mm class medium. We assume that $u_2 = 0$ and all other quantities depend only on x_1 and x_3. The vibration regime is considered steady according to

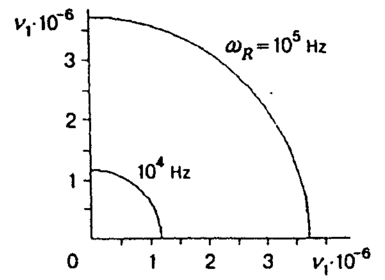

Fig. 3

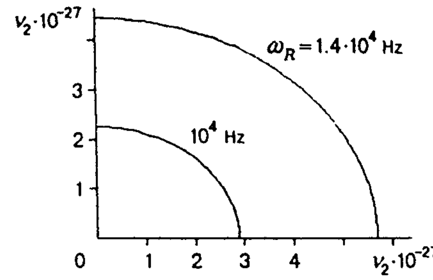

Fig. 4

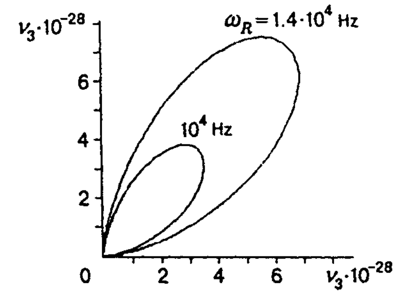

Fig. 5

the law $\exp[-i\omega t]$. In this case L_{ij} is written as

$$L_{11} = c_{11}\partial_1^2 + c_{44}\partial_3^2 + \rho\omega^2, \quad L_{12} = L_{21} = (c_{13} + c_{44})\partial_1\partial_3, \quad L_{13} = L_{31} = (e_{31} + e_{15})\partial_1\partial_3,$$
$$L_{14} = -\gamma_{11}\partial_1, \quad L_{22} = c_{44}\partial_1^2 + c_{33}\partial_3^2 + \rho\omega^2, \quad L_{23} = L_{32} = e_{15}\partial_1^2 + e_{33}\partial_3^2, \quad L_{24} = -\gamma_{33}\partial_3,$$
$$L_{42} = -i\omega T_0\gamma_{33}\partial_3, \quad L_{33} = -(\ni_{11}\partial_1^2 + \ni_{33}\partial_3^2), \quad L_{34} = g_3\partial_3, \quad L_{41} = -i\omega T_0\gamma_{11}\partial_1,$$
$$L_{43} = i\omega T_0 g_3\partial_3, \quad L_{44} = -i\omega\rho c_\epsilon - k_{11}\partial_1^2 - k_{33}\partial_3^2,$$

and the vector of unknowns $\mathbf{U}$ as $\mathbf{U} = \{u_1, u_3, \varphi, \theta\}$.

The functions $\Psi_j^{(m)}(\mathbf{x}, \boldsymbol{\xi})$ that satisfy the system of equations $L_{ij}\Psi_j^{(m)} + \delta_{im}\delta(\mathbf{x}, \boldsymbol{\xi}) = 0$ and vanish at infinity are considered fundamental solutions. Using Fourier transform, we can easily construct integral representations for $\Psi_j^{(m)}$:

$$\Psi_j^{(m)}(\mathbf{x}, \boldsymbol{\xi}) = \frac{1}{(2\pi)^2} \int\limits_{R_2} \frac{P_{jm}(\boldsymbol{\alpha}, \omega)}{P_0(\boldsymbol{\alpha}, \omega)} \exp[i(\boldsymbol{\alpha}, \boldsymbol{\xi} - \mathbf{x})]d\boldsymbol{\alpha}. \tag{2.1}$$

Here $P_{jm}(\boldsymbol{\alpha}, \omega)$ and $P_0(\boldsymbol{\alpha}, \omega)$ are polynomials in $\boldsymbol{\alpha}$ and ω; and $\boldsymbol{\alpha} = (\alpha_1, \alpha_3)$. We note that representation (2.1) is not very suitable for practical applications, and, therefore, we shall simplify it by analysis of the integrands and contour integration.

It should be noted that the polynomials $P_{jm}(\alpha_1, \alpha_3, \omega)$ in (2.1) are of different orders (in contrast to the "pure" electroelasticity problem in [12]), i.e., some of them are polynomials of the 6th order in $\boldsymbol{\alpha}$ and the remaining polynomials are of the 5th order. According to such sign, we divide these polynomials into two types: 1) polynomials of the 6th order and 2) polynomials of the 5th order. The first type polynomials possess the evenness property $P_{jm}(-\alpha_1, -\alpha_3, \omega) = P_{jm}(\alpha_1, \alpha_3, \omega)$, and while the second type polynomials the oddness property $P_{jm}(-\alpha_1, -\alpha_3, \omega) = -P_{jm}(\alpha_1, \alpha_3, \omega)$. The polynomials P_{14}, P_{24}, P_{34}, and those obtained by permutation of indices are related to the second type, and the remaining polynomials to the first type.

To simplify (2.1) for the first type we represent the integrand in the form

$$\frac{P_{jm}(\alpha\cos\psi,\alpha\sin\psi,\omega)}{P_0(\alpha\cos\psi,\alpha\sin\psi,\omega)}=\sum_{k=0}^{3}\frac{a_{jmk}(\psi,\omega)}{\alpha^2-\zeta_k^2(\psi,\omega)}$$

$[\zeta_0=0,\ a_{jmk}(\pi+\psi,\omega)=a_{jmk}(\psi,\omega)$, and $\zeta_k^2(\pi+\psi,\omega)=\zeta_k^2(\psi,\omega)=\omega^2 z_k^2]$ and set $|\mathbf{x}-\boldsymbol{\xi}|=r$, $\cos\psi_1=(\xi_1-x_1)/r$, and $\sin\psi_1=(\xi_3-x_3)/r$. Then, we have

$$\Psi_j^{(m)}=\frac{1}{(2\pi)^2}\int_0^\infty\int_0^{2\pi}\sum_{k=0}^{3}\frac{a_{jmk}(\psi,\omega)}{\alpha^2-\zeta_k^2(\psi,\omega)}\exp\left(i\alpha r\cos(\psi-\psi_1)\right)\alpha\,d\alpha\,d\psi$$

$$=\frac{1}{(2\pi)^2}\int_0^\infty\int_0^{\pi}\sum_{k=0}^{3}\frac{a_{jmk}(\psi,\omega)}{\alpha^2-\zeta_k^2(\psi,\omega)}\left[\exp\left(i\alpha r\cos(\psi-\psi_1)\right)+\exp\left(-i\alpha r\cos(\psi-\psi_1)\right)\right]\alpha\,d\alpha\,d\psi.$$

Let us consider the integral

$$\mathrm{I}_2(z,\zeta)=\int_0^\infty\frac{\exp(i\alpha z)+\exp(-i\alpha z)}{\alpha^2-\zeta^2}\alpha\,d\alpha,\quad \operatorname{Im}\zeta>0,\quad \operatorname{Re}\zeta>0.$$

To evaluate it we introduce the contours

$$C_\Gamma^+=[0,R]\bigcup C_R^+\bigcup[iR,0]\quad\text{and}\quad C_\Gamma^-=[0,R]\bigcup C_R^-\bigcup[-iR,0]$$

(C_R^+ and C_R^- are parts of the circumference with radius R with its center at the coordinate origin which lie in the first and fourth quadrants, respectively).

Using the contour integration and Jordan's lemma [13], we obtain

$$\mathrm{I}_2(z,\zeta)=\pi i\exp(i\zeta z)+2\int_0^\infty\frac{\exp(-\tau z)}{\tau^2+\zeta^2}\tau\,d\tau.$$

We write the latter integral in the form

$$\mathrm{I}_2(z,\zeta)=\pi i\exp(i\zeta z)-2[\operatorname{cosi}(\zeta z)\cos(\zeta z)+\operatorname{sini}(\zeta z)\sin(\zeta z)],$$

where $\operatorname{cosi}(x)$ and $\operatorname{sini}(x)$ are the integral cosine and sine.

For $z<0$, $\mathrm{I}_2(z,\zeta)$ is evaluated similarly. Uniting these two cases, we have the following representation for the fundamental solutions:

$$\Psi_j^{(m)}=\frac{1}{(2\pi)^2}\int_0^{\pi}\sum_{k=0}^{3}a_{jmk}(\psi,\omega)\mathrm{F}_2(\zeta_k(\psi,\omega)|r\cos(\psi-\psi_1)|)\,d\psi,\tag{2.2}$$

where $\mathrm{F}_2(z)=(\pi i/2)\exp(iz)-\operatorname{cosi}(z)\cos(z)-\operatorname{sini}(z)\sin(z)$.

Remarks. (1) $\mathrm{F}_2(\zeta_0)$ denotes the limiting value of the function $\mathrm{F}_2(z)$ as $z\to 0$ with accuracy to constants that are insignificant from the viewpoint of construction of the fundamental solution, i.e., $\mathrm{F}_2(\zeta_0 r|\cos(\psi-\psi_1)|)=-\ln|r\cos(\psi-\psi_1)|$.

(2) Since only $P_{33}(0,\psi,\omega)\neq 0$ of the 6th order polynomials and the relation $P_{jm}(\alpha,\psi,\omega)=\alpha^2 P_{jm}^*(\alpha,\psi,\omega)$ is valid for the remaining [$P_{jm}^*(\alpha,\psi,\omega)$ are 4th order polynomials], we have all $a_{jm0}=0$ except for a_{330}, and the representation (2.2) can be written as

$$\Psi_j^{(m)}=\frac{1}{(2\pi)^2}\int_0^{\pi}a_{330}(\psi,\omega)\ln|r\cos(\psi-\psi_1)|\,d\psi\,(\delta_{3j}\delta_{3m})$$

$$+\frac{1}{(2\pi)^2}\int_0^{\pi}\sum_{k=1}^{3}a_{jmk}(\psi,\omega)\mathrm{F}_2(\zeta_k(\psi,\omega)|r\cos(\psi-\psi_1)|)\,d\psi.\tag{2.3}$$

We simplify representation (2.1) for polynomials of the second type. We take into account that the representation $P_{jm}(\alpha, \psi, \omega) = -i\alpha P^*_{jm}(\alpha, \psi, \omega)$ is valid for polynomials of the second type, where $P^*_{jm}(\alpha, \psi, \omega)$ are polynomials of the 4th order. In this case, we have

$$\frac{P_{jm}(\alpha\cos\psi, \alpha\sin\psi, \omega)}{P_0(\alpha\cos\psi, \alpha\sin\psi, \omega)} = -i\alpha^{-1}\sum_{k=1}^{3}\frac{b_{jmk}(\psi, \omega)}{\alpha^2 - \zeta_k^2(\psi, \omega)},$$

where $b_{jmk}(\pi + \psi, \omega) = -b_{jmk}(\psi, \omega)$.

Proceeding similarly and introducing the function $\mathrm{F}_1(z) = (\pi/2)\exp(iz) - (\mathrm{cosi}(z)\sin(z) - \mathrm{sini}(z)\cos(z))$, where $z > 0$, we finally obtain

$$\Psi_j^{(m)} = \frac{1}{(2\pi)^2}\int_0^\pi \sum_{k=1}^{3}\frac{b_{jmk}(\psi, \omega)}{\zeta_k(\psi, \omega)}\mathrm{F}_1(\zeta_k(\psi, \omega)|r\cos(\psi - \psi_1)|)\mathrm{sgn}(\cos(\psi - \psi_1))\,d\psi \qquad (2.4)$$

($j = 4$; $m = 1$, 2, and 3; $m = 4$; and $j = 1$, 2, and 3).

Integral representations (2.3) and (2.4) make it possible to apply effectively the boundary element method to thermoelectroelasticity problems. They have characteristic logarithmic singularities intrinsic for two-dimensional problems.

This work was supported by the Russian Foundation for Fundamental Research (Grant 94-01-01259).

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